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DISCUSSION OF NUMERICAL COMPUTATION OF BUCKLING LOADS BY FINITE DIFFERENCES

(*Published in December, 1949*)

By I. K. Silverman, Bruno A. Boley, G. S. Ramaswamy, Chi-Teh Wang, George Herrmann, and Mario G. Salvadori

STRUCTURAL DIVISION

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DISCUSSION

I. K. SILVERMAN,[27] ASSOC. M. ASCE.—In this very interesting paper Mr. Salvadori has shown that the solution of a differential equation is actually the simultaneous solution of an infinite number of algebraic equations and, conversely, for many practical cases, the simultaneous solution of a finite number of algebraic equations may be accepted in lieu of the solution of a differential equation. The practical consequences of this conclusion are apparent and Mr. Salvadori has solved some problems that arise in the theory of structures. The Rayleigh energy method,[2] widely used for the approximate solution of problems in elastic stability and vibrations, depends on the assumption of the deflected shape of the structure. There are some cases (the curved bar, for example) in which the shape of the deflected bar cannot be estimated and the solution of the differential equation is required.

The absence of any physical explanation of the method of finite differences is somewhat detrimental to a full appreciation and understanding of the method. Physically, the method of finite differences is equivalent to substituting a less complex structure for the ideal mathematical structure. In 1920 Heinrich Hencky[28,29] proposed that, in studying the elastic stability of structures, the continuous elastic structure be replaced by one consisting of rigid sections connected with hinges capable of resisting elastic deformation. For example, the column in Fig. 11(*a*) is to be replaced by Fig. 13, in which parts AB, BC,

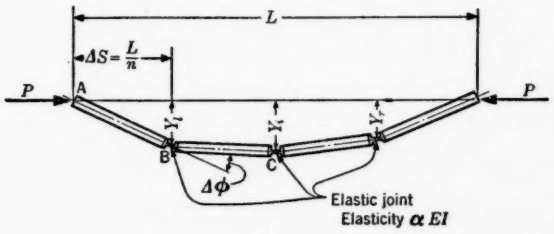


FIG. 13

etc., are considered rigid and the elasticity is concentrated at the hinges. From a study of this model the designer can decide whether the substitute structure is more rigid or less rigid than the original continuous elastic structure, and if the critical load is expected to be greater than or less than that for the continuous

NOTE.—This paper by Mario G. Salvadori was published in December, 1949, *Proceedings*. The numbering of footnotes, equations, tables and illustrations in this Separate is a continuation of the consecutive numbering used in the original paper.

[27] Engr., U. S. Bureau of Reclamation, Denver, Colo.

[2] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936.

[28] "Über Die Angenäherte Lösung Von Stabilitätsproblemen Im Raum Mittels Der Elastischen Gelenkkette," by Heinrich Hencky, *Der Eisenbau*, Vol. 11, 1920, pp. 437–452.

[29] "Drang Und Zwang," by August Föppl and Ludwig Föppl, R. Oldenbourg, München and Berlin, Germany, 1920, Vol. 2, p. 368.

elastic structure. The designer is also interested in knowing if an approximate solution gives values on the safe side or the unsafe side. The problem is thus no longer in the realm of pure mathematics and enters the domain of physics.

That Mr. Salvadori's application of finite first order differences is mathematically identical with Mr. Hencky's "bar chain" method is apparent from a comparison of the mathematical computations involved. The bar chain method for the problem of Fig. 11(*a*) furnishes a set of linear algebraic equations by expressing the relation between curvature and moment. Thus,

$$\frac{1}{\rho} = \frac{\Delta\phi}{\Delta s} = \frac{n\,\Delta\phi}{L} = \left(\frac{n}{L}\right)^2 (2\,y_i - y_l - y_r) = \frac{M}{E\,I} = \frac{P\,y_i}{E\,I} \quad \ldots\ldots (79a)$$

and

$$2\,y_i - y_l - y_r = \frac{P\,L^2}{n^2\,E\,I}\,y_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (79b)$$

The following determinant is obtained—

$$\begin{vmatrix} (2-w) & -1 & 0 & \ldots. & 0 & 0 & 0 \\ -1 & (2-w) & -1 & \ldots. & 0 & 0 & 0 \\ 0 & -1 & (2-w) & \ldots. & 0 & 0 & 0 \\ \ldots. & \ldots. & \ldots. & & \ldots. & \ldots. & \ldots. \\ \ldots. & \ldots. & \ldots. & & \ldots. & \ldots. & \ldots. \\ \ldots. & \ldots. & \ldots. & & \ldots. & \ldots. & \ldots. \\ 0 & 0 & 0 & \ldots. & (2-w) & -1 & 0 \\ 0 & 0 & 0 & \ldots. & -1 & (2-w) & -1 \\ 0 & 0 & 0 & \ldots. & 0 & -1 & (2-w) \end{vmatrix} = 0 \quad \ldots (80)$$

in which

$$w = \frac{P\,L^2}{n^2\,E\,I} = b_n + 2 = \frac{k}{n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (81)$$

The determinantal equations become

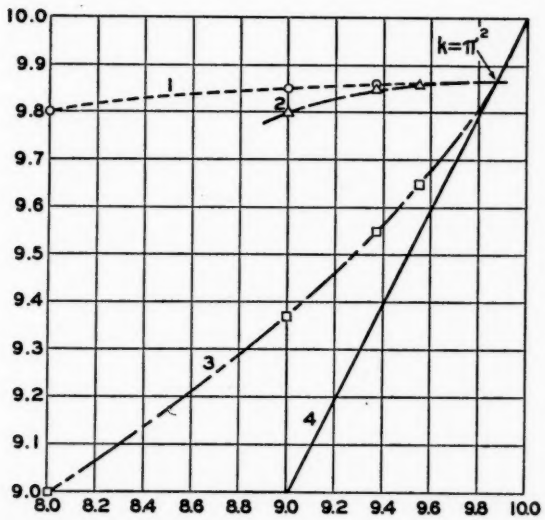
For n	Equations	w	k
2	$(2-w) = 0$	2	8
3	$(2-w)^2 - 1 = 0$	1	9
4	$(2-w)^3 - (2-w) - (2-w) = 0$	$2 - \sqrt{2}$	9.37258

Further applications[28] have been made to torsional equilibrium, curved bars, and to the elastic equilibrium of frames. A point to be emphasized is that the solution for the substitute structure is as "exact" as the "exact" solution of the mathematical continuous structure and it only remains for the designer to decide whether the substitute structure is sufficiently similar to the actual structure for his purposes. From the results given in the paper it can be inferred that the bar chain column subjected to axial loads is a less rigid structure than the continuously elastic column and that results obtained are on the safe side. On the other hand, for case *C* (Section 9) the bar chain beam is less rigid or more rigid depending on whether elastic resistance is concentrated in an odd or even number of hinges. The behavior of k is thus tied in with the stiffness of the substitute structure compared with the ideal mathematical structure.

BRUNO A. BOLEY.[30]—A clear presentation of the finite differences approach, and a very logical development of a valuable method for buckling load calculations, are given in this paper. One of the chief advantages of methods such as this is that, although sometimes the mathematics is lengthy, no complicated mathematical manipulation is required which might overshadow the physical significance of the various steps. In fact, once the governing differential equation for a particular problem is established, the buckling load may be obtained with operations not more complicated than those of algebra.

TABLE 14.—VARIATION OF k WITH n

n	2	3	4	5	6
k_n.....	8.000	9.000	9.373	9.549	9.646
$k_{n,n+1}$....	9.800	9.852	9.863	9.867	9.868



CURVE	ABSCISSA	ORDINATE	SYMBOL
1	k_n	$k_{n,n+1}$	----○----
2	k_{n+1}	$k_{n,n+1}$	——△——
3	k_n	k_{n+1}	—-□-——
4	ABSCISSA = ORDINATE		——

FIG. 14.—EXAMPLE OF GRAPHICAL EXTRAPOLATION

The method suggested is useful when it is possible, without too much difficulty, to write a differential equation (or at least a system of differential equations) governing the problem under consideration, and the corresponding boundary conditions. In some cases, however, it is difficult to obtain such a system of equations; examples of such problems are the buckling of reinforced plates or shells, or the instability of frameworks. In these problems some other methods probably could be more easily applied—in particular, some of the methods developed during the investigations in 1947 at the Polytechnic Institute of Brooklyn,[31] in Brooklyn, N. Y. These methods are also finite difference methods in the sense that the structure under investigation is considered to be composed of several units of finite dimensions.

[30] Eng. Specialist, Goodyear Aircraft Corp., Akron, Ohio.

[31] "Numerical Methods for the Calculation of Elastic Instability," by Bruno A. Boley, *Journal of the Aeronautical Sciences*, June, 1947, p. 337.

The usefulness of the method suggested by Mr. Salvadori is greatly increased, of course, by the extrapolation procedures suggested in the paper. The examples in the paper show that these procedures provide a very powerful tool. Their chief drawback is (as Mr. Salvadori points out) that it is not possible to know beforehand whether the value they yield for the buckling load is higher or lower than reality. It may be mentioned here that the methods referred to previously[31] will yield an exact answer if the analysis of the units is sufficiently rigorous. In general, the units may be chosen sufficiently simple so that an accurate analysis is possible. If an assumption is made (as those methods allow) concerning the deflected shape of the structure, it may be shown that the resulting calculated buckling load will be too high.

To obtain a better idea of the accuracy afforded by the extrapolation procedures, it is necessary that additional problems be solved involving different types of structures. In any one problem, however, a good estimate of the correct buckling load may probably be obtained by some graphical method of extrapolation, such as the rather obvious one (for the case of the Euler buckling of a straight column) outlined herein. The values in Table 14 were given by Mr. Salvadori in which $k_n = \frac{P_{cr}}{E\,I/L^2}$ for the case in which the bar is assumed divided into n equal segments, and $k_{n,\,n+1}$ is the value obtained by the h^2-extrapolation—that is,

$$k_{n,\,n+1} = \frac{(n+1)^2\,k_{n+1} - n^2\,k_n}{(n+1)^2 - n^2} \qquad (82)$$

As n becomes very large, the values of k_n and $k_{n,\,n+1}$ approach the correct value, k; in this case $k = \pi^2 = 9.8696$. In fact, at the limit, it may be stated that

$$k_n = k_{n,\,n+1} = k_{n+1} \qquad (83)$$

Thus, if plots are made of k_n against $k_{n,\,n+1}$, k_{n+1} against $k_{n,\,n+1}$, and k_n against k_{n+1}, the common intersection of the three resulting curves will represent the correct value, k. Furthermore, this intersection will occur on a line on which the abscissa is equal to the ordinate. These plots are given in Fig. 14 for the present example.

A number of such schemes may be devised either to improve the accuracy of the method or to estimate the error introduced by considering the structure comprising a finite number of points. In many cases, however, the extrapolation procedure leads to such good results that no improvement may be required for engineering accuracy.

G. S. RAMASWAMY,[32] JUN. ASCE.—The use of difference equations to obtain approximate solutions to boundary value problems in science and engineering is not new;[33] but the author's clear exposition is certain to make the method familiar to engineers.

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[32] Prof. of Civ. Eng., Annamalai Univ., Annamalainagar, South India.

[33] "Mathematical Theory of Elasticity," by I. S. Sokolinkoff, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1946.

The writer has undertaken to check the validity of the extrapolation procedures discussed by the author and to study the effect of varying boundary conditions on convergence. The example chosen is the classical Euler problem, the strut being hinged at one end and restrained in position and direction at the other end. The exact value[34] of P_{cr} for this case is $20.16\ E\ I/L^2$. Table 15 shows the computations based on first order differences.

TABLE 15.—CHECK COMPUTATIONS

n	$\frac{P_n L^2}{E I}$	e_n (%)	h^2-extrapolations	
2	12.00	40.5	2, 3	19.31304
3	16.0628	20.4	3, 4	19.9421
4	17.760	11.9	4, 5	20.1722
5	18.614	7.72	....	....

It is seen that the mode of approach in this case is monotonic. Comparison with Example A of Mr. Salvadori's paper reveals that convergence is very much slower for the boundary conditions assumed by the writer. However, h^2-extrapolations give surprisingly accurate results even with very few trials.

CHI-TEH WANG.[35]—The power of the Richardson method[10] of extrapolation in the computation of buckling loads is demonstrated in this paper. Although the solution of elastic stability problems by finite differences has been a subject of much study by many eminent workers, without the use of the method of extrapolation, the computations become very tedious when an accurate solution is required. In applying the method of extrapolation, Mr. Richardson assumed that the error in the buckling load is also of the h^2-type which the author has argued as not necessarily true. However, by solving many numerical examples, Mr. Salvadori has shown that the assumption did work; but, because of the lack of proof, he had to limit his conclusion by adding the phrase "of experimental characteristic." Nevertheless, the fact that the assumption did work out so well in all these cases appears more than a mere coincidence.

Mr. Richardson's assumption can actually be proved to be rigorous at least for the cases where the moment of inertia is constant. Consider the case of the buckling of a column with hinged ends under axial compression. The differential equation is

$$\frac{d^2Y}{dx^2} + K\,Y = 0 \qquad (84)$$

for which the boundary conditions are $Y = 0$ at $x = 0$ and $x = L$. The corresponding difference problem is

$$y\,(x + h) - 2\,y\,(x) + y\,(x - h) + k\,h^2\,y\,(x) = 0 \qquad (85)$$

and $y = 0$ at $x = 0$ and $x = L$. In Eq. 85, y is the approximate value of Y and k is the approximate value of K obtained by solving the difference equation (Eq. 85) rather than the differential equation (Eq. 84).

[34] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1936, p. 89.

[35] Associate Prof. of Aeronautical Eng., Daniel Guggenheim School of Aeronautics, College of Eng., New York Univ., New York, N. Y.

[10] "The Approximate Arithmetical Solution by Finite Differences of Physical Problems Involving Differential Equations with an Application to the Stresses in a Masonry Dam," by L. F. Richardson, *Philosophical Transactions*, Royal Soc. of London, Vol. 210, 1911.

The finite difference equation (Eq. 85) can be solved by assuming the solution to be of the form:

$$y = c\, e^{mx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (86)$$

Substituting Eq. 86 in Eq. 85,

$$c\, e^{mx+mh} - 2\, c\, e^{mx} + c\, e^{mx-mh} + c\, k\, h^2\, e^{mx} = 0 \quad \ldots\ldots\ldots (87)$$

or

$$[(e^{mh} + e^{-mh}) + (k\, h^2 - 2)]\, c\, e^{mx} = 0 \quad \ldots\ldots\ldots\ldots (88)$$

The form of Eq. 88 requires that

$$(e^{mh} + e^{-mh}) + (k\, h^2 - 2) = 0 \quad \ldots\ldots\ldots\ldots\ldots (89)$$

or

$$\cosh m\, h + \left(\frac{k\, h^2}{2} - 1\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots (90)$$

Three forms of the solution for m may be obtained depending on whether $\left|\frac{k\, h^2}{2} - 1\right| > 1$, $\left|\frac{k\, h^2}{2} - 1\right| = 1$, or $\left|\frac{k\, h^2}{2} - 1\right| < 1$. In the first two cases, it can be shown that the resulting solution for y cannot satisfy the boundary conditions and therefore that these cases will not be considered. When $\left|\frac{k\, h^2}{2} - 1\right| < 1$, $\cosh m\, h < 1$ and m is a pure imaginary number. Writing $m = i\, \lambda$, Eq. 90 becomes

$$\cos \lambda\, h = 1 - \frac{k\, h^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (91)$$

therefore,

$$\lambda = \pm \frac{1}{h} \cos^{-1}\left(1 - \frac{k\, h^2}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots (92)$$

Substituting values of λ from Eq. 92 in Eq. 86, the values of k can then be determined from the boundary conditions. As this procedure proves rather laborious, the solution is simplified by first determining the value of λ from the boundary conditions and then solving k from Eq. 91. Since λ can be either positive or negative, the general solution for y is

$$y = c_1 \sin \lambda\, x + c_2 \cos \lambda\, x \quad \ldots\ldots\ldots\ldots\ldots (93)$$

To satisfy the boundary conditions, it is necessary that

$$\lambda = \frac{n\, \pi}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (94)$$

in which n is any integer, and $n = 1$ at the instant of buckling. Since $1 - \cos 2\, \theta = 2 \sin^2 \theta$, Eq. 93 can be rewritten, substituting the value of λ obtained from Eq. 94; thus:

$$k \left(\frac{h}{2}\right)^2 = \sin^2 \frac{\pi\, h}{L\, 2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (95)$$

Since $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$, expanding the sine function in Eq. 95 yields

$$k\left(\frac{h}{2}\right)^2 = \left[\frac{\pi}{L}\frac{h}{2} - \frac{1}{3!}\left(\frac{\pi}{L}\frac{h}{2}\right)^3 + \cdots\right]^2$$

$$= \frac{\pi^2}{L^2}\left(\frac{h}{2}\right)^2 - \frac{2}{3!}\left(\frac{\pi}{L}\right)^4\left(\frac{h}{2}\right)^4 + \cdots \quad (96)$$

Dividing Eq. 96 by $\left(\frac{h}{2}\right)^2$,

$$k = \frac{\pi^2}{L^2} - \frac{2}{3!}\left(\frac{\pi}{L}\right)^4\left(\frac{h}{2}\right)^2 + \cdots = K - \alpha_1 h^2 - \alpha_2 h^4 + \cdots \quad (97)$$

Eq. 97 results from the fact that $K = \frac{P}{E\,I} = \frac{\pi^2}{L^2}$, thus proving that

$$e = K - k = \alpha_1 h^2 + \alpha_2 h^4 + \cdots \quad (98)$$

is of the h^2-type. It may be noted that for columns with other boundary conditions, if the differential equation is of the form as given by Eq. 84, Eq. 93 shows that λ is always equal to $\sqrt{K}$ and therefore the proof holds equally true in these cases. In general, if the governing differential equations are those with constant coefficients, the corresponding finite difference equations can be solved by the same symbolic method as in the case of the differential equations and the final solutions for y have the same form as those for Y. Therefore λ will always be equal to $\sqrt{K}$ and the relation given by Eq. 98 can always be proved to be correct.

GEORGE HERRMANN.[36]—The consideration of stability problems in different fields of engineering is steadily becoming of greater importance. The value of this paper lies in two facts: (1) The finite differences method (which had not been thoroughly investigated) is presented in a very broad and clear manner; and (2) through his simple extrapolation formulas, the author increases the power of the method to a large extent.

The worker engaged in the investigation of characteristic value problems will be thankful to the author for considering so many aspects of the method (such as different coordinate systems, different order of operators, procedures for numerical computations, types of asymptotic approach, etc.), and will notice that the method does not require any guessing of the deflection line (or surface), such as is required by the energy method and other approximate methods.[37] Since complicated boundary conditions (for example, elastically supported edges[38]) can be expressed by finite differences, the method is applicable to a larger variety of problems than indicated by the paper. The extrapolation formulas show an amazing power, as can be seen by the many illustrative examples. By their use the numerical computations are made less cumbersome, with the saving of time and energy.

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[36] Instr. in Civ. Eng., Columbia Univ., New York, N. Y.

[37] "Berechnung der Beulspannungen gedrückter Rechteckplatten," by F. Stüssi, *Abhandlungen*, Internationale Vereinigung für Brückenbau und Hochbau, Zürich, Bd. 8, 1947, p. 237.

[38] "Knickung der rechteckigen Platte bei veränderlicher Randbelastung," by Shizuo Ban, *ibid.*, Zürich, Bd. 3, 1935, p. 1.

There are, however, many problems of elastic stability in which the energy method leads to good results in a shorter time than any other approach. For example,[39,40,41,42] the plate in Fig. 15 is exposed to a thrust of triangular shape along the simply supported edges b. The load-free edges, a, are built in and free, respectively. Apply the energy method, assuming a sinusoidal deflection function in the direction of the thrust and a deflection function of the type—

$$Y_n = A\left(\cos\frac{P_n y}{b} + \cosh\frac{P_n y}{b}\right) + B\left(\cos\frac{P_n y}{b} - \cosh\frac{P_n y}{b}\right) + C\left(\sin\frac{P_n y}{b} + \sinh\frac{P_n y}{b}\right) + D\left(\sin\frac{P_n y}{b} - \sinh\frac{P_n y}{b}\right) \quad (99)$$

in the direction perpendicular to the thrust. For the ratio $a/b = 1.635$, Eq. 99 yields $k = 1.636$ in the first approximation and practically the same value in the second approximation, the difference being noticeable only in the fifth digit. Using the finite differences method with first order operators, $k_2 = 1.14$ for $n = 2$, and $k_4 = 1.35$ for $n = 4$. With the extrapolation formula (Eq. 41), $k_{2,4} = 1.42$; the error is still 13% and to find k_4 it was necessary to solve a fourth degree equation.

Although this example is not typical, as far as convergence is concerned, in general, the investigator dealing with stability of plates, is interested in finding the ratio a/b of the edges of the plate for which k is minimum. The buckling equation being established, it is, in general, easier to find k (minimum) by the energy method, because the ratio a/b appears only in the diagonal members of the determinant, whereas, in using finite differences, this ratio appears in most elements of the determinant.

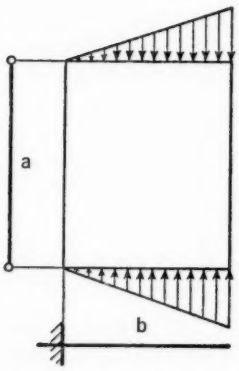

FIG. 15

The extrapolation formulas can be applied in solving not only the characteristic value problems, but also the boundary (or initial) value problems. For a given problem the type of approach depends on the choice of the type of coordinates. Thus in the problem of a skew plate with equal sides 2 a, under uniform load p, and simply supported along all four edges, the true value for the deflection can be squeezed, using different types of coordinates.

Using skew coordinates for the analysis of a 60° skew plate, H. Favre[13] obtained a central deflection of $f_2 = 0.0351\,\frac{a^4 p}{D}$ for $n = 2$, and $f_4 = 0.0385\,\frac{a^4 p}{D}$

[39] "Elastische Beulung von auf einseitigen, ungleichmässigen Druck beanspruchten Platten," by C. F. Kollbrunner and G. Herrmann, *Mitteilung No. 1*, Technische Kommission des Verbandes Schweizerischer Brückenbau- und Stahlhochbau-Unternehmungen, Verlag Leemann, Zürich, 1948.

[40] "Reine Biegungsbeulung rechteckiger Platten in elastischen Bereich," by C. F. Kollbrunner and G. Herrmann, *Mitteilung No. 2*, Technische Kommission des Verbandes Schweizerischer Brückenbau-und Stahlhochbau-Unternehmungen, Verlag Leemann, Zürich, 1949.

[41] *Schweizerische Bauzeitung*, March 13, 1948, p. 146.

[42] *Ibid.*, May 28, 1949.

[13] "Le calcul des plaques obliques par la méthode des équations aux differences," by H. Favre, Mémoires, Association Internationale des Ponts et Charpentes, Zürich, Vol. 7, 1943–1944.

for $n = 4$, D being the flexural rigidity of the plate. Applying the extrapolation formula probably a better value $f_{2,4} = 0.0396 \frac{a^4 p}{D}$ results.

Using triangular coordinates, F. L. Ehasz[43] computed the values $f_4 = 0.0453 \frac{a^4 p}{D}$ for $n = 4$, and $f_8 = 0.0419 \frac{a^4 p}{D}$ for $n = 8$, which have been rounded off here to three figures. In this case the extrapolation formula (Eq. 41) gives $f_{4,8} = 0.0408 \frac{a^4 p}{D}$. The average of the two extrapolated values is $0.0402 \frac{a^4 p}{D}$ and, plausibly, does not differ more than approximately 1% from the true value of the deflection of the center. This example illustrates, at the same time, how important it is to choose an appropriate type of coordinates, as emphasized by the author.

Extremely good approximations are obtained, using extrapolation formulas, in computing the deflections of uniformly loaded square plates. In the case of simple supports,[13] $f_2 = 0.06250 \frac{a^4 p}{D}$ and $f_4 = 0.06445 \frac{a^4 p}{D}$. The extrapolated value (using Eq. 41) equals $f_{2,4} = 0.06510 \frac{a^4 p}{D}$ and differs by only 0.2% from the exact value $f = 0.06496 \frac{a^4 p}{D}$. In the case of built-in edges,[44] $f_4 = 0.00180 \frac{a^4 p}{D}$ and $f_8 = 0.00143 \frac{a^4 p}{D}$. In spite of the fact that these values differ from the exact value $\left(f = 0.00128 \frac{a^4 p}{D}\right)$ by 41% and 12%, respectively, the extrapolation formula (Eq. 41) leads to $f_{4,8} = 0.00131 \frac{a^4 p}{D}$, which is only 2.3% in error.

In the case of transverse concentrated loads applied to a plate, the extrapolation formulas given by the author cannot be used, because the error is due not only to the substitution of finite differences for differentials, but also to the spreading of the load over a certain area.[45] In investigating skew plates under a central concentrated load,[46] the writer had to use higher order differences to obtain satisfactory approximations. It might be worthwhile to investigate the possibility of developing extrapolation formulas to problems of this type.

MARIO G. SALVADORI,[47] ASSOC. M. ASCE.—Mr. Silverman points out that the Hencky "bar chain" method is equivalent to the finite differences approach and he emphasizes the advantages of a physical picture in solving engineering problems. The choice between a physical or a mathematical

[43] "Structural Skew Plates," by F. L. Ehasz, *Transactions*, ASCE, Vol. 111, 1946, p. 1011.

[44] "Die elastischen Platten," by Arpád Nádai, J. Springer, Berlin, 1925, p. 220.

[45] "Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten," by H. Marcus, J. Springer, Berlin, Vol. 1, 2d Ed., 1932, p. 61.

[46] "Theoretische Untersuchungen über die Durchbiegung parallelogrammförmiger Platten unter zentrischer Einzellast," by G. Herrmann, *Publication No. 4*, Laboratoire de Photoélasticité de l'École Polytechnique Féderale, Verlag Leemann, Zürich, 1950.

[47] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y.

approach rests essentially on the mental attitude of the worker and on the relative efficiency of the two types of approach. In this connection it will suffice to note that the physical approach of Mr. Hencky would never lead to the investigation of the error and to the discovery of the extrapolations, without which his method is excessively laborious and inefficient. The problems dealt with in the paper illustrate the decided superiority of the mathematical approach in the present case. In other cases, a combination of physical intuition and mathematical knowledge usually gives the best results.

The graphical "extrapolation of the extrapolations," suggested by Mr. Boley, although without sound mathematical justification, may prove useful. It can be applied to any converging series of approximate values and has the advantage of bringing the asymptotic value of the approximations to a point on a 45° line. It is doubtful whether the graphical extrapolation would give accurate results in the case of oscillating approximations.

Mr. Ramaswamy's example, which checks an unpublished result obtained by the writer, clearly indicates the powerfulness of the extrapolation procedure. It is interesting to add that the $(h^2 - h^4)$-extrapolation applied to the approximations $n = 2, 3, 4$, gives (in this case) $k = 20.166$, with an error of 0.03%.

The writer is grateful to Mr. Herrmann for showing the applicability of the extrapolations to problems other than those treated in the paper. The extrapolations can actually be used in connection with any method of successive approximations in which: (*a*) The successive values of the unknown depend on an increasing integer n and (*b*) the error is expandable into a series of negative even powers of n. Various procedures of this type, to which the extrapolations can be usefully applied have been discussed elsewhere by the writer.[48]

The proof, given by Mr. Wang, that the error in the characteristic value of Euler's problem is of the h^2-type represents an important contribution to the theory of the finite differences method. It opens the path to a more general proof which is valid for all ordinary equations with constant coefficients.

The writer is unable to follow the reasoning in the last paragraph of Mr. Wang's discussion, however, and he does not see that it proves Eq. 98 to be correct for all equations with constant coefficients. In fact, Eq. 98 was derived for the particular case of Eq. 84 and this equation does not hold even for all beams with constant moment of inertia. The following fundamental theorem may instead be proved, by the error-formula (Eq. 34) given by the writer:

> *Theorem.* The error in the characteristic value of linear ordinary differential equations with constant coefficients evaluated by finite differences is of the h^2-type.

For this purpose notice that, according to the theorem of L. F. Richardson, e_1 and e_2, and hence $e_1 + k\,e_2$, are of the h^2-type. Therefore, the condition:

$$L_1\,y + K\,L_2\,y = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (100)$$

is a sufficient condition (although not a necessary condition) for e to be of the h^2-type; but the solution y of an ordinary difference problem with constant

[48] "Approximations and Extrapolations," by M. G. Salvadori, *Columbia Engineering Quarterly*, January, 1949.

coefficients is identical with solution Y of the corresponding differential problem[49] and, hence, by Eq. 30 (the condition of Eq. 100), is satisfied when L_1 and L_2 have constant coefficients. Thus, for equations with constant coefficients the error e reduces to the form:

$$e = -\frac{e_1 + K\, e_2}{L_2\, Y - e_2} \ldots\ldots\ldots\ldots\ldots\ldots (101)$$

and is of the h^2-type.

It is now obvious that the extrapolations must give good results in connection with constant coefficient equations and that the accuracy of the extrapolations must improve by taking into account a larger number of terms in the series for the error. For example, the writer has found that, on a large number of cases, the $(h^2 - h^4)$-extrapolation gives better results than the h^2-extrapolation in connection with constant coefficient equations.

It is also interesting to note that the behavior of the successive approximations is monotonic in all the constant coefficient examples solved by the writer, whereas the behavior is seldom monotonic for equations with variable coefficients. For this last type of equation the sufficient condition of Eq. 100, in general, is not satisfied; and this fact may be proved by checking the values of the approximate solution at the pivotal points, obtained by finite differences, to note that they differ from the values of the integral of the differential equation at the same points. For example, the differential problem for the buckling of a strut, built in at one end and free at the other, under the action of its own weight q, may be written in the form:

$$\frac{d^2u}{dz^2} + k^2\, z\, u = 0 \ldots\ldots\ldots\ldots\ldots\ldots (102)$$

and

$$u'\,(0) = u\,(L) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (103)$$

and has a series solution:

$$u\,(z) = C\left(1 - \frac{k^2\, z^3}{2\cdot 3} + \frac{k^4\, z^6}{2\cdot 3\cdot 5\cdot 6} - \frac{k^6\, z^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right) \ldots\ldots (104)$$

(S. Timoshenko[2] develops Eq. 102; here $u = dy/dx$, $z = L - x$, and $k^2 = q/(E\,I)$.) By dividing the strut into two equal parts, the ratio of the middle point, slope $u_1 = u\,(L/2)$, to the end slope, $u_2 = u\,(0)$, becomes (with $k^2\, L^3 = 7.81$ and, hence, $k^2\,(L/2)^3 = 0.977$): $u_1/u_2 = 0.842/1 = 0.842$. The corresponding difference problem is given by the equations:

$$u_l + \left(\frac{k^2\, L^2}{n^2}\, z_i - 2\right) u_i + u_r = 0 \ldots\ldots\ldots\ldots\ldots (105)$$

and

$$u'\,(0) = u(L) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (106)$$

For $n = 2$, Eq. 105 and Eq. 106 give rise to the system of two equations:

$$2\, u_1 - 2\, u_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (107)$$

[49] "Die gewöhnlichen und partiellen Differenzengleichungen der Baustatik," by Fr. Bleich and E. Melan, Springer, Berlin, 1927.

and

$$\left(\frac{k^2 L^3}{8} - 2\right) u_1 + u_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (108)$$

The root of the corresponding determinantal equation is $k^2 L^3 = 8$ ($e = 2.4\%$); and, with this value for k, the ratio u_1/u_2 becomes

$$\frac{u_1}{u_2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (109)$$

Hence, the approximate u differs from the true u and Eq. 100 is not necessarily satisfied.

Finally, it must be noted that, for equations with variable coefficients, an increase in the number of terms of the error series taken into account usually reduces the accuracy of the extrapolations obtained by means of all the successive approximations. Therefore, these results seem to indicate that the error in the characteristic values of equations with variable coefficients is not of the h^2-type.

The reason why the extrapolations work so well in connection with any monotonic series of approximations remains to be investigated, although this result could well be of a purely empirical character.

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